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THE PROBLEM OF TIME-OPTIMAL CONTROL WITH SEARCH FOR A TARGET POINT*

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A formulation of the problem of guaranteed time-optimal control is given such that the position of the target point (the right-hand end of the trajectory) is defined to within some set of uncertainty. The objective of the control is to detect (to observe) the target point within the boundary of this set and to bring the phase vector of the controlled system to the target point as quickly as possible. The target point is assumed to be known (detected) exactly if it belongs to an information region that moves along with the phase vector. In the case where the controlled system performs a simple motion in a plane, the set of uncertainty is a bounded convex domain and the information region is a half-plane, an algorithm for constructing the optimal phase pattern of trajectories searching for the target point is proposed and substantiated. Examples are discussed. The paper develops the investigations of /1/, and the subject matter of the article is close to that of /2, 3/.

1. Formulation of the problem. A two-point problem of programmed time-optimal control for a dynamical system is given by the following relations:

$$x^{\cdot} = f(x, u, t),\ u \in U,\ t \in [t_0, T]$$
$$x(t_0) = x^0,\ x(T) = x^1,\ J = T - t_0 - \min \tag{1.1}$$

Here $x \in R^n$ is the phase vector, u and U are the control vector and the set of admissible values of u, and x^0 and x^1 are the initial and final values of the phase vector. In the classical formulation /4/ the admissible programmed control is constructed on the basis of the complete information about relations (1.1), i.e. about the dynamics of the system and about the vectors $x^0,\ x^1 \in R^n$, which is available to the controlling side. Such a control is defined as a function $u = u(x^0, t_0, x^1; t),\ t \in [t_0, T]$, where x^0, t_0, x^1 play the role of parameters. In the present paper it is assumed that the exact position of the target point x^1 is unknown to the controlling side at the initial instant.

We shall specify more precisely the information that is available to the controlling side and we shall give a description of the class of admissible control functions. At the initial instant, apart from the dynamical relations (1.1) and the position x^0, t_0, the controlling side knows the set of uncertainty D such that

$$x^1 \in D,\ D \subset R^n \tag{1.2}$$

In order to bring the system to the point $x(T) = x^1$, the controlling side should be able to obtain information that is more accurate than (1.2). Such a possibility is described mathematically with the aid of a moving information region $G = G(x(t))$ that depends on the phase vector of the system, where $G(x) = \{\xi \in R^n\colon\ \xi - x \in G_0\}$, with $G_0 = G(0)$ being a given set. Therefore, $G(x(t))$ is a fixed domain in the moving coordinate system with origin at $x(t)$ and with axes parallel to the axes of the initial coordinate system. By assumption, $x^1 \notin G(x^0)$. The exact value of the vector x^1 becomes known to the controlling side at the first instant $t = t_* \geqslant t_0$ when the observation (detection) condition

$$x^1 \in G(x(t))\ (t = t_*) \tag{1.3}$$

is met (Fig.1).

Thus, at any instant during the motion along the trajectory $x(t)$ corresponding to some control function $u(t)$, the controlling side has access to the values of $t, x(t)$, and information on whether inclusion (1.3) is satisfied or not, and in the case when the inclusion is satisfied they also have access to exact information about x^1.

We shall assume that D and G are closed simply connected domains. In addition, in order that, generally speaking, a finite time T should be realized, we assume that D is bounded. Although the convexity of the sets is not discussed as a special case, in practical situations

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the most characteristic domains D and G are convex.

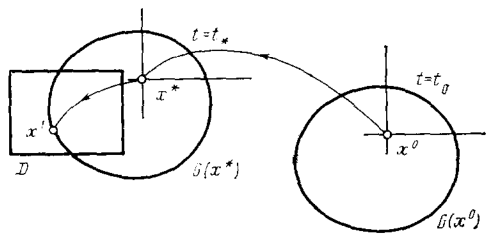

Fig.1

We introduce into our disucssion the following sets in R^n:

$$X_0 = \{x \in R^n: D \subset G(x)\},\ X_1 = \{x \in R^n: G(x) \cap D \neq \varnothing\} \tag{1.4}$$

It is obvious that X_0 is contained in X_1. If $x(t) \in X_0$, then it is guaranteed that x^1 will be observed at a time t. If $x(t) \in X_1$, then the detection at time t is possible but not certain. We shall assume that x^0 belongs to the domain $X = R^n \setminus X_1$. This means that the target point cannot be detected at the initial instant. Generally speaking, the cases where D and $G(x^0)$ have common points but x^1 is not observed, i.e. $x^0 \notin X_1$ and $x^1 \notin G(x^0)$, are also interesting. In such cases, by replacing D by a slightly contracted set, one can reduce the discussion to the previous case.

Let us describe the class of admissible control functions. Under the given assumptions concerning information about the system, it is obvious that the final condition $x(T) = x^1$ must not be guaranteed in the class of purely programmed control functions. It is necessary to correct (to change) the programmed control function after obtaining information about x^1 at the instant t_* when condition (1.3) is satisfied. In this connection, we shall assume that piecewise programmed control functions consisting of two parts given by a pair of control functions

$$u = \{u_0(x^0;\ t),\ u_1(x^*,\ t_*,\ x^1;\ t)\} \tag{1.5}$$

with values in U are admissible. Here x^0, x^1, x^*, t_* play the role of parameters with values in the domains $x^0 \in X, x^* \in X_1, x^1 \in D,\ t_* \geqslant t_0$. The other parameters t_0, D and G_0 which define the functions are omitted in our notation for u_0. With respect to the time variable, the functions u_i are defined in sufficiently long intervals $t_0 \leqslant t \leqslant \vartheta_0,\ t_* \leqslant t \leqslant \vartheta_*$. Only general requirements are imposed on the character of the dependence on time and upon the properties of the functions in (1.1) so as to make sure that there is a unique solution of system (1.1) in the time interval in question.

Therefore, to construct a control function of the form (1.5), it is necessary to obtain exact information on the parameters x^*, t_* and x^1 for the motion at the instant t_* in addition to the initial information mentioned above (Fig.1).

The motion of system (1.1) corresponding to a control function of the form (1.5) can be constructed in the following way. Eq.(1.1) with the control function $u_0(x^0;\ t)$ for $t \geqslant t_0$ is integrated up to the time t_* when condition (1.3) is satisfied and the vector x^1 becomes known. Next, the control function $u_1(x^*,\ t_*,\ x^1;\ t)$ for $t_* \leqslant t \leqslant T$ is used, which ensures that the equality $x(T) = x^1$ is satisfied at the time T. Only those pairs (1.5) are regarded as being admissible for which the corresponding values of t_* and T are finite. We shall assume that the set of admissible control functions is non-empty for all vectors x^0 in question.

If x^0 is fixed, then for each admissible control function (1.5) and for each vector $x^1 \in D$, there is a corresponding trajectory of system (1.1) passing through x^1 and there is the corresponding time of motion $J(x^0, x^1, u) = T - t_0$.

Let us consider the problem of the guaranteed minimum time of approaching x^1.

Problem 1. Find the guaranteed minimum time of action $J_*(x^0)$ and the admissible control function u^* that provides the minimum

$$J_*(x^0) = \min_u \max_{x^1} J(x^0, x^1, u) = \max_{x^1} J(x^0, x^1, u^*),\quad x^1 \in D \tag{1.6}$$

Here minimization is carried out over all admissible control functions. It is assumed that the extreme values are attained.

It is fairly obvious that the second component of the control function u^* of the form (1.5) that appears in (1.6) is the time-optimal programme of action for problem (1.1) with full information and with initial position x^*, t_*. We shall assume that this component is, in principle, known. Then the control function $u_0(x^0; t)$ for the search problem and the corresponding trajectory constitute the basic object sought.

As far as information in the system is concerned, the above controlled system with incomplete information is a model for some control problems in robot technology /1/ and also for a number of other control and search problems.

An obvious illustration of the proposed model is given by identifying the information region $G(x(t))$ with a cone of light that can be moved around in a "dark" space to illuminate (to detect) a stationary object x^1 in a given domain D.

The basic difference between the formulation of the problem posed in the present paper and the problems of guaranteed search /2, 3/ lies in the fact that apart from finding the stationary object x^1, one should also bring the phase vector to x^1 as quickly as possible. Moreover, in the case of search problems it is usually assumed that $G(x) \subset D$ for all x, i.e., the search is carried out inside the domain D. Below, we consider the case where X_0 is a non-empty set, i.e., there are vectors $x \in R^n$ such that $D \subset G(x)$.

Let us note a simple consequence of (1.6):

$$J_*(x^0, D') \leqslant J(x^0, D),\ D' \subset D \tag{1.7}$$

where the corresponding set D is included in the notation for J_*.

2. ***The problem in a plane.*** Let $n = 2$ and let system (1.1) have the simple form

$$\begin{gathered} z^{\cdot} = u,\ |u| \leqslant 1,\ z(t_0) = z^0,\ z(T) = z^1 \\ z = (x, y) \in R^2,\ |u| = \sqrt{u_1^2 + u_2^2} \end{gathered} \tag{2.1}$$

D is an arbitrary bounded domain with doubly smooth boundary and G is a right half-plane whose boundary is parallel to the ordinate axis with the phase vector belonging to the boundary of the stationary information domain. Then each of the sets X_0 and X_1 given by (1.4) is a left half-plane whose boundary is tangent to D. In particular, ∂X_0 coincides with the ordinate axis (Fig.2).

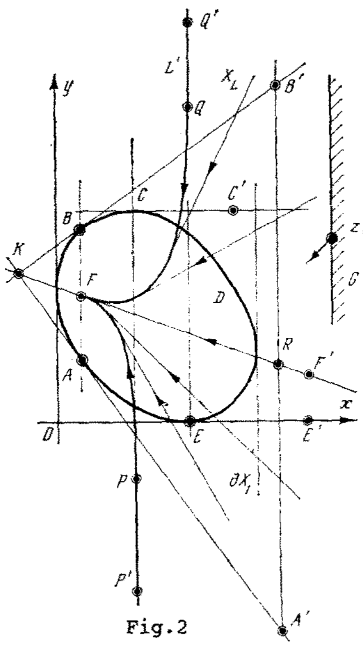

Fig.2

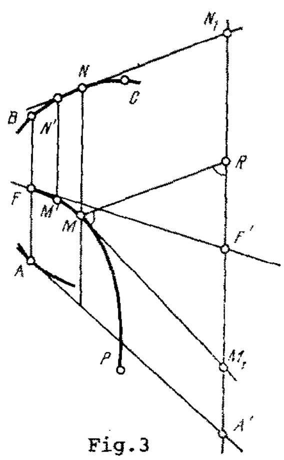

Fig.3

As a preliminary, we construct a field of trajectories of system (2.1) in the domain $X = R^2 \setminus X_1$. It will be shown later that the field corresponds to the optimal control function $u_0(z^0; t)$ for the search problem. The necessary illustrations are given in Fig.2. We choose an orthogonal system of coordinates xy such that D is contained in the first quadrant and the coordinate axes are tangent to D. We construct a vertical interval AB (i.e., an interval that is parallel to the y axis) so that AB is the chord that is closest to the y axis and such that the lines tangent to D at the end-points of the chord are perpendicular to each other (and intersect each other at K). Such an interval exists and is unique since, by virtue of the smoothness of the boundary ∂D, there is a continuous monotonic one-to-one function $\beta(x)$ defined in some interval $0 \leqslant x \leqslant x'$, $\beta(0) = \pi$, $\beta(x') = 0$, where $\beta(x)$ is the angle between the lines tangent to ∂D at the points of intersection of ∂D and the vertical line with the first coordinate $x \in [0, x']$. The point $F(x_F, y_F)$, which is called the focal point, is the mid-point of the interval AB. The lines tangent to D at C and E are horizontal. PP' and QQ' are vertical lines passing through C and E, respectively. The convex (concave) curve FQ (FP) represents the graph of a function $y = y(x)$, where

$$y(x) = y_F + \int_{x_F}^{x} \frac{\varphi'^2(\tau) - 1}{2\varphi'(\tau)} d\tau \tag{2.2}$$

The function $y = \varphi(x)$ defines a convex (concave) segment AE (BC) of the boundary of D. As follows from an analysis of the derivative of (2.2), the curves FQ and FP are connected smoothly with the rays QQ' and PP'. The convexity properties of these curves follow from the formula $y'' = {}^1/_2\varphi''(1 + 1/\varphi'^2)$, which is obtained by differentiating (2.2) and from the fact that D is a convex domain. If integral (2.2) over the left half-neighbourhood of the point C (E) diverges, then the point P (Q) is an infinitely distant point. The straight line FF' is a common tangent of the curves FQ and FP at F.

We denote by L the piecewise smooth curve PFQ with a cusp at F and without the end-points P and Q, and we denote by L' the curve $Q'QFPP'$, whose branches tend to infinity. We denote by X_L the open domain at the right-hand side of L'. It is obvious that $X \subset X_L$. We consider the family of rays tangent to L, (i.e. half-lines cut off from the tangent lines by the points of tangency) that run into the domain X_L.

Since the derivative $y'(x)$ is a monotonic function, for each point in X_L there is exactly one ray from the given family passing through this point. The motion of a phase point with the maximum unit speed from z^0 along the ray to the curve L, then along L to the focal point F, and further along the straight line FK to the coordinate axis, will be referred to as the reference motion.

Theorem. The reference motion of a point $z^0 \in X$ is a motion along the optimal trajectory of the search problem corresponding to the control function $u_0^*(z^0; t)$.

Proof. First, we remark that the last segment of the trajectory leading to z^1 is a vertical interval, for z^1 turns out to lie on the boundary of the half-plane G at the instant of detection t_*, and the time-optimal motion from z^* to z^1 is the motion along the straight line that connects the points.

First, we shall prove that the motion along the ray $F'F$ is optimal. We consider the two-point set $D' = \{A, B\}$. It can be shown that for any point $R \in X_L$, minimax (1.6) with $D = D'$ is equal to the common length of the broken lines RFA and RFB. Here and in what follows we shall refer to the length of a path rather than to the time of motion, for the phase point moves with unit velocity. We assert that if $R \in FF'$, then (1.7) becomes an equality. We take a vertical line $A'B'$ passing through R. By construction, the interval KR is a median of the triangle $A'KB'$ and its length is equal to that of each of the intervals RB' and RA' and the broken lines RFB and RFA. Since KB' and KA' are tangent to the convex figure D, the target point can turn out to be in the interval $A'B'$ only and the distance from the target point to R cannot exceed the length of the interval RB'. The worst positions of the target are the points A and B, for which the given time of reaching the target is exact. In this way it is established that the basic motion along $F'F$ is optimal and the guaranteed time is found. In particular, if the initial point R coincides with F, then the guaranteed time is equal to the common length $|FB|$ of the intervals FA and FB.

We shall prove that each initial point $M \in L'$ has a similar property, namely the guaranteed time is equal to the length of the interval of the vertical line from M to the corresponding point N that belongs to the curve BC (or AE if $M \in FQ$ see Fig.3). Since z^1 may coincide with N, it is obvious that the guaranteed time of motion $J_*(M)$ with M being the initial point cannot be less than the length of the interval MN. For the guaranteed time $J_*(M)$ to remain the same for the motion towards F along the arc PF with z^1 to be found at N', say, the lengths of the paths MN and $MM'N'$ should be equal. We can write this condition in the form

$$\varphi(x + \Delta x) - y(x + \Delta x) + \Delta s = \varphi(x) - y(x) \tag{2.3}$$

where $(x, y(x))$ and $(x + \Delta x, y(x + \Delta x))$ are the coordinates of M and M', and Δs is the length of the arc MM'.

By letting Δx in (2.3) tend to zero, one can obtain the differential relation

$$y'(x) = \frac{\varphi'^2(x) - 1}{2\varphi'(x)} \tag{2.4}$$

Formula (2.2) is obtained by integrating this relation.

It is easy to convince oneself that for any initial point P' on the vertical ray PP' (Fig.2) the guaranteed time is equal to the length of the interval $P'C$ and that this is guaranteed by the motion along the vertical line up to the point P, and then (unless z^1 turns out to be in the interval PC) along the reference trajectory. One more estimate is necessary to show that the result $|MN|$ cannot become worse (greater) if z^1 turns out to lie on the vertical line MN below M. The estimate follows from a more general estimate obtained at the end of the proof.

Hence we have established that for each initial point belonging to the curve L', the reference motion is optimal, and we have found the corresponding time of motion. We note that for the entire domain D to be surveyed (for the detection to be guaranteed), any trajectory of the search problem starting in the right half-plane $x > 0$ should end on the ordinate axis, and so the trajectory should intersect L'. According to what was shown, after reaching the curve L', the search should be carried out along the reference trajectory. In this connection, we consider the problem of the optimal motion of system (2.1) from $z^0 \in X_L$ to the line L' with the integral part of the functional equal to the time of motion along L' and with the terminal part equal to $J_*(z')$, where $z' \in L'$ and where J_* is the guaranteed time of bringing the system from z' and z^1, which is already known:

$$\dot{z} = u,\ |u| \leqslant 1,\ z(0) = z^0 \in X_L,\ z(\theta) \in L' \qquad (2.5)$$
$$J = \theta + J_*(z(\theta)) - \min$$

Problem (2.5) can be solved by the methods of /4/ or by direct geometric methods. The set of optimal trajectories of (2.5) coincides exactly with the set of lines tangent to the curves FQ and FP. It is obvious that the optimal value of the functional $J'(z^0)$ corresponding to (2.5) represents a lower estimate for the time $J_*(z^0)$:

$$J'(z^0) \leqslant J_*(z^0),\ z^0 \in X_L \qquad (2.6)$$

We shall prove that the times in (2.6) are equal. To do this, we only need to prove that for the motion along a line tangent to L, the target point cannot be detected from a distance greater than the length of the remaining path of the search problem. Indeed, for an arbitrary point M_1, the angles at the vertices M and R of the triangle M_1MR corresponding to $MN = RN_1$ turn out to be equal to each other owing to relation (2.4) between the tangent functions of the slopes of the tangent lines MM_1 and NN_1. Then both the length of the interval M_1N_1 and the length of the broken line M_1MN are equal to $J'(M_1)$. Since D is convex, the distance to the tangent line is not less than the distance to any other point of D that lies above the straight line FF'. For the points below the line FF', the estimate can be obtained with the use of the tangent line AA'. The required estimate

$$|M_1M| + |MF| + |FA| \geqslant |M_1A'|$$

follows from the equality $|F'M_1| + |M_1A'| = |FF'| + |FA|$, which was stated at the beginning of the proof as a property of the straight line FF', from the obvious estimate $|FM_1| \leqslant |M_1M| + |FM|$, and from the triangle inequality for $FF'M_1$. Here $|FM|$ is the length of the arc FM, while the other lengths are understood to be those of the corresponding intervals.

In a similar way one can discuss the case of lines tangent to FQ. Thus, the proof of the theorem is completed.

We remark that the constructed field of trajectories forms only two segments of the boundary of D, namely BC and AE. In particular, the convex hull of the two arcs represents the smallest set D that generates the field. The same field is suitable for any bounded convex domain D lying inside the region $C'CBKAEE'$ such that the boundary of D contains the arcs BC and AE. For a (bounded) domain D stretching itself as far to the right as desired, all the tangents of the curves FQ and FP (without the right end-points) enter and fill the half-plane X.

The above proof can be generalized to the case where it is required merely that D be a bounded convex domain. By Theorem 2 of /5, p.477/ the properties of convexity and monotonicity used in the proof are satisfied. Another method of constructing the field of trajectories of the search problem for a domain D whose boundary is not smooth consists in majorizing D by some domain $D' \supset D$ with smooth boundary and finding the limiting field as D' is contracted to D. In particular, in the case where D is an interval, the trajectories of the search problem constructed in this way agree with the results of /1/, where such a field was constructed by another method.

3. ***The minimum interval of observation.*** We assume that checking whether the detection condition (1.3) is satisfied is connected with the loss of some resources, and the controlling side is interested in starting the observation as late as possible. It is obvious that if the point $x(t)$ is sufficiently far from D so that the intersection $D \cap G(x(t))$ is empty, than

there is no need to carry out the observation (to switch a sensing device on). The appropriate moment to switch it on is when the vector $x(t)$ becomes an element of the set X_1 given by (1.4). However, nothing will be lost as far as the guaranteed time of bringing the system to the target point is concerned if the sensing device is not switched on for some additional time after intersecting the boundary of X_1.

Indeed, let $S(x^0, x^1)$ be the time of the optimal fast motion for system (1.1) with the complete information. We will take some position $x(t) \in X_1$ as the initial position x^0 at $t = t_0$ and we will assume that before the arrival at x^0 the sensing device has not been switched on. If now, at the instant t_0, the sensing device is switched on, then it may turn out that x^1 is already inside the domain of observation. The longest time of motion towards the point x^1 detected in this way is equal to the maximum value

$$S^*(x^0) = \max_{x^1} S(x^0, x^1), \quad x^1 \in D \cap G(x^0) \tag{3.1}$$

It is obvious that as long as the time $S^*(x^0)$ does not exceed $J_*(x^0)$, there is no need to switch the sensing device on. The inequality

$$S^*(x^0) \leqslant J_*(x^0) \tag{3.2}$$

defines a domain in the space of x and t such that one can fail to carry out the observation while moving inside this domain without any loss as far as the original guaranteed result is concerned. For the automatic problem, inequality (3.2) defines a domain in the space of x. In the formulation of the problem given in /1/, the domain $Y \subset R^n$ of motion with the sensing device switched off was within the discretion of the controlling side and was considered as a control element, i.e., the instant of switching the sensing device on was determined from the position (from the observation of the phase vector). The largest domain Y corresponds to the case where the times in (3.2) are equal.

4. *Examples.* We consider the problem of Sect.2 with the boundary of D being an ellipse $(x-a)^2/a^2 + y^2/b^2 = 1$. The focal point F (which does not coincide with the foci of the ellipse) belongs to the abscissa axis. Computations lead to the following value of the x-coordinate of F: $x_F = a - a^2/\sqrt{a^2+b^2}$. The arcs BC and AE are given by the relations $\varphi(x) = \pm b/a \cdot \sqrt{a^2 - (x-a)^2}$. Equality (2.2), which defines the envelopes FQ and FP, has the form

$$\pm y(\xi) = \frac{a^2+b^2}{2ab}\sqrt{a^2-\xi^2} - \frac{\sqrt{a^2+b^2}}{2} - \frac{a^2}{2b}\ln\frac{a(a+\sqrt{a^2-\xi^2})}{\xi(b+\sqrt{a^2+b^2})}, \qquad \xi = a - x$$

The points Q and P are infinitely distant points. The phase pattern of the trajectories of the search problem for the case of a circle $(a = b = \rho)$ is shown in Fig.4. The optimal trajectories of the search problem are distinct from the straight motion towards the centre of the circle and contain curvilinear segments.

Let D be a quadrangle with sides a and with centre at the point $(a/2, 0)$ (see Fig.5). It follows from the estimates of Sect.2 that the optimal trajectories of the search problem will remain the same if the vertical interval AB is taken as D. Using the remark made at the end of Sect.2 or the results obtained in /1/, one can show that the optimal trajectories of the search problem are the straight lines directed towards the mid-point F of AB, just as in the case where D is equal to the interval AB.

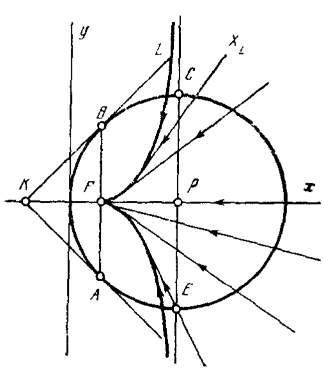

Fig.4

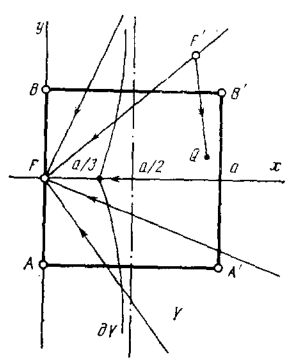

Fig.5

Let us consider the problem of Sect.3. For the case of the simple motion in question, the boundary of the region Y defined by (3.2) consists of points F' such that

$$|FF'| = \max_{Q} |F'Q|, \quad Q \in D \cap G(F')$$

The maximum is, of course, attained either at C or at E. Computations lead to the following equality defining the boundary of Y (Fig.5):

$$x = {}^1/_3 \, [2\,(a + |y|) - \sqrt{a^2 + 2a\,|y| + 4y^2}], \quad -\infty < y < +\infty$$

Therefore, the sensing device that checks whether condition (1.3) is satisfied does not need to be switched on until the constructed curve is reached. As one can see in Fig.5, the sensing device is switched on after the domain G has "swept" more than a half of the quadrangle D.

We remark that the above constructions are suitable in the case where D consists of four points only, namely of the vertices of the quadrangle. The points A and B define the field of trajectories and the points C and E define the domain Y.

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